Bicharacteristics Based Numerical Modelling for Hyperbolic Systems in Several Variables

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First order non-linear equation, existence of characteristic curves and compatibility conditions

\[ F(x, y, u, p, q) = 0, \quad p = u_x, \quad q = u_y \]

Charpit equation splits into characteristic equations or \textit{kinematic equations}

\[ \frac{dx}{d\sigma} = F_p, \quad \frac{dy}{d\sigma} = F_q \]

and compatibility conditions or \textit{dynamical equations}

\[ \frac{dp}{d\sigma} = -F_x - pf_u \]
\[ \frac{dq}{d\sigma} = -F_y - qf_u \]
\[ \frac{du}{d\sigma} = pf_p + qf_q \]
Simplest Hyperbolic Equation in Multi-D

The FO NLPDE is an example of a hyperbolic equation. But more:

- A first order linear PDE for $u : \mathbb{R}^{m+1} \to \mathbb{R}$ is

  $$a(x, t)u_t + \langle b(x, t), \nabla_x u \rangle = c(x, t)$$

  where $a : \mathbb{R}^{m+1} \to \mathbb{R}$, $b : \mathbb{R}^{m+1} \to \mathbb{R}^m$, $c : \mathbb{R}^{m+1} \to \mathbb{R}$.

- Characteristic PDE for the characteristic surface: $\varphi(x, t) = 0$ in $(x, t)$-space is

  $$a(x, t)\varphi_t + \langle b(x, t), \nabla_x \varphi \rangle = 0.$$

- Characteristic surface is generated by characteristic curves

  $$\frac{d\mathbf{x}}{d\sigma} = b(x, t), \quad \frac{dt}{d\sigma} = a(x, t).$$
Hyperbolic system in one space variable

- Compare with single FO linear PDE and write the system of linear FO equations:
  \[ A(x, t)u_t + B(x, t)u_x + C(x, t) = 0 \]
  where \( A, B \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^n \).

- The system is defined to be hyperbolic if
  (i) the matrix \( B \) has \( n \) real eigenvalues relative to the matrix \( A \), i.e.,
  the \( n \) roots of the equation in \( \lambda \)
  \[ \det(A\varphi_t + B\varphi_x) = 0 \quad \text{or} \quad \det(B - \lambda A) = 0, \quad \lambda = -\frac{\varphi_t}{\varphi_x} \]
  are real and
  (ii) the dimension of the eigenspace of each eigenvalue is equal to its algebraic multiplicity.
Hyperbolic system in one space variable: example

- \( u_{1t} + cu_{2x} = 0, \quad u_{2t} + cu_{1x} = 0, \quad c = \text{constant} \), which leads to the wave equation \( u_{1tt} - c^2 u_{1xx} = 0 \).
- \( \implies \) vector form of equations
  \[
  \begin{bmatrix}
  1 & 0 \\
  0 & 1 \\
  \end{bmatrix}
  \begin{bmatrix}
  u_1 \\
  u_2 \\
  \end{bmatrix}_t
  +
  \begin{bmatrix}
  0 & c \\
  c & 0 \\
  \end{bmatrix}
  \begin{bmatrix}
  u_1 \\
  u_2 \\
  \end{bmatrix}_x
  = 0.
  \]

- The Characteristic equation is
  \[
  \det(B - \lambda A) \equiv \lambda^2 - c^2 = 0 \implies \lambda = -c, \quad c.
  \]
- Let \( \ell_1 \) be the left eigenvector of \(-c\) i.e.,

  \[
  \ell_1 \begin{bmatrix}
  -\lambda & c \\
  c & -\lambda \\
  \end{bmatrix} \equiv \ell_1 \begin{bmatrix}
  c & c \\
  c & c \\
  \end{bmatrix} = 0.
  \]
Hyperbolic system in one space variable: example ··· conti.

- $\ell_1 = [1, -1]$. Similarly $\ell_2 = [1, 1]$.
- Multiply the vector form of the equations by $\ell_1$ and $\ell_2$ and get the two independent compatibility conditions:
  \[
  (u_1 - u_2)_t - c(u_1 - u_2)_x = 0
  \]
  and
  \[
  (u_1 + u_2)_t + c(u_1 + u_2)_x = 0
  \]

- **Definition:** $(u_1 - u_2)$ and $(u_1 + u_2)$ are characteristic variables of the first and second characteristic family (also Reimann Invariants of the second and first characteristic family).
Hyperbolic system in one space variable ··· conti.

- Denote eigenvalues by $c_1, c_2, \ldots, c_n$ and assume that
  \[ c_1 \leq c_2 \leq c_3 \leq \ldots \leq c_n \]

- Denote the left and right eigenvectors corresponding to $c_i$ by $\ell^{(i)}$ and $r^{(i)}$, respectively:
  \[ \ell^{(i)}(B - c_iA) = 0 \quad \text{and} \quad (B - c_iA)r^{(i)} = 0, \quad i = 1, 2, \ldots, n. \]

- **Characteristic curves** in $(x, t)$-plane corresponding to the eigenvalue $c_i$ are given by the equation
  \[ \frac{dx}{dt} = c_i : \quad \text{or} \quad \varphi_t + c_i \varphi_x = 0. \]
Multiplying the FO system by $\ell^{(i)}$, we get the compatibility condition along a characteristic curve of the $i$th characteristic family

$$\ell^{(i)} A (\partial_t + c_i \partial_x) u + \ell^{(i)} c = 0$$

which means the dynamical equation

$$\ell^{(i)} A \frac{du}{dt} + \ell^{(i)} c = 0 \quad \text{along} \quad \frac{dx}{dt} = c_i.$$

Note that if $c_i$ is a multiple eigenvalue of multiplicity $p$ and there exists $p$ linearly independent eigenvectors corresponding to this eigenvalue and we get $p$ independent compatibility conditions corresponding to it.
A First Order Quasilinear System in multi-Dimensions

- We shall consider now a system of \( n \) first order quasilinear PDE in \( m + 1 \) dimensional \((x, t)\)-space:

\[
A u_t + B^{(\alpha)} u_{x\alpha} + C = 0, \tag{1}
\]

where a repeated Greek letter index represent sum over the range 1, 2, \ldots, \( m \) and \( u \in \mathbb{R}^n \) and \( A, B^{(\alpha)} \in \mathbb{R}^{n\times n} \) and \( C \) are functions of \( x, t \) and \( u \).

- In general the system is quasilinear, i.e., \( A, B^{(\alpha)} \) are functions of \( u \) also.

- For a first order quasilinear hyperbolic system, the kinematical and dynamical results get coupled due to genuine nonlinearity present in a mode of propagation.
The characteristic PDE of (21) is

\[ Q(x, t; \nabla \varphi, \varphi_t) := \det(A \varphi_t + B^{(\alpha)} \varphi_{x\alpha}) = 0. \]

We may assume that the system (1) is hyperbolic, i.e., it has \( n \) real eigenvalues and its eigenspace is complete. However, we shall concentrate on just one eigenvalue \( c \) and assume:

- \( c \) simple eigenvalue satisfying

\[ Q(x, t; \nabla \varphi, \varphi_t) = 0 \Rightarrow \det(n_{\alpha} B^{(\alpha)} - cA) = 0. \] (2)
Example

For the wave equation

\[ u_{tt} - a_0^2 (u_{xx} + u_{yy}) = 0, \]  

the characteristic PDE is

\[ \phi_t - a_0 (\phi_x^2 + \phi_y^2)^{1/2} = 0. \]

An important solution is the characteristic conoid

\[ t - t_0 \pm \frac{1}{a_0} \left( (x - x_0)^2 + (y - y_0)^2 \right)^{1/2} = 0 \]  

Figure: Characteristic conoid
A Hyperbolic System of First Order Quasilinear Equations

... continued

- For the characteristic surface $\Omega : \varphi(x, t) = \alpha$:

$$c = -\frac{\varphi_t}{|\nabla \varphi|}, \quad \mathbf{n} = \frac{\nabla \varphi}{|\nabla \varphi|},$$

(5)

$c = \text{velocity of the wavefront } \Omega_t,$

$n = \text{unit normal of } \Omega_t.$

- Let $\ell$ and $r$ be the corresponding left and right eigenvectors

$$\ell(n_\alpha B^{(\alpha)} - cA) = 0, \quad (n_\alpha B^{(\alpha)} - cA)r = 0.$$  

(6)
Theorem Part A: **Kinematics.** For the eigenvalue $c$ the rays are given by

$$\frac{d x_\alpha}{dt} = (\ell B^{(\alpha)} r)/(lA r) = \chi_\alpha$$  \hspace{1cm} (7)

which is the lemma on bicharacteristic (C & H) and

$$\frac{d n_\alpha}{dt} = -\frac{1}{\ell A r} \ell \left\{ n_\beta \left( n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta_\beta^\alpha} - c \frac{\partial A}{\partial \eta_\beta^\alpha} \right) \right\} r$$  \hspace{1cm} (8)

where $u$ is present in (6) and (7) also through $\ell$ and $r$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_\alpha \frac{\partial}{\partial x_\alpha}, \hspace{1cm} \frac{\partial}{\partial \eta_\beta^\alpha} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta}$$  \hspace{1cm} (9)

is a tangential derivative in the wavefront $\Omega_t$. 

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**Bicharacteristic Theorem Part A, Prasad 1993, 2007**
Theorem Part B: **Dynamics**. Further, the system (2) implies a compatibility condition on a characteristic surface $\Omega$ of hyperbolic system in the form

$$\ell A \frac{du}{dt} + \ell (B^{(\alpha)} - \chi_\alpha A) \frac{\partial u}{\partial x_\alpha} + \ell C = 0 \quad (10)$$

The operator

$$\tilde{\partial}_j = l_i (B^{(\alpha)}_{ij} - \chi_\alpha A_{ij}) \frac{\partial}{\partial x_\alpha} \quad (11)$$

is a tangential derivative not only on the characteristic surface $\Omega$ but also on the wavefront $\Omega_t$. 
Proof of Bicharacteristic Theorem

- Proof of (7) follows geometrically from lemma of bicharacteristics of (C & H) or algebraically from (5) Prasad, 2007. It is further proved that $\chi$ is indeed a ray rector.

- Proof of (8) follows from the Charpit’s equations of (5) and converting the equation for $\varphi_{x_\alpha}$ into the equation for $n_\alpha$ using (3) (see Prasad, 2007, Ind Jour Pure and Appl Math).

- For proof of (10), we premultiply (1) by $\ell$ and rearranging the terms (Prasad, Nonlinear Hyperbolic Waves in Multi-dimensions, Chapman and Hall/CRC, 121 (2001)).
Bicharacteristic formulation of the Euler equations

We consider the 3-D Euler equations of gas dynamics

\[ \rho_t + \langle q, \nabla \rho \rangle + \rho \langle \nabla, q \rangle = 0, \]  
\[ q_t + \langle q, \nabla \rangle q + \frac{1}{\rho} \nabla p = 0, \]  
\[ p_t + \langle q, \nabla \rangle p + \rho a^2 \langle \nabla, q \rangle = 0, \]  

where \( \rho \) is the mass density, \( q = (q_1, q_2, q_3) \) the fluid velocity, \( p \) the pressure, \( a \) is sound velocity in the medium given by

\[ a^2 = \gamma p / \rho \]

and \( \gamma \) is the ratio of specific heats, assumed to be constant. It is a system of 5 first order quasilinear equations.
Euler equations contd.

Taking \( u = (\rho, q_1, q_2, q_3, p)^T \), we find \( A = I \) and the matrix

\[
B^{(\alpha)} = \begin{bmatrix}
q_\alpha & \rho \delta_1 \alpha & \rho \delta_2 \alpha & \rho \delta_3 \alpha & 0 \\
0 & q_\alpha & 0 & 0 & \rho^{-1} \delta_1 \alpha \\
0 & 0 & q_\alpha & 0 & \rho^{-1} \delta_2 \alpha \\
0 & 0 & 0 & q_\alpha & \rho^{-1} \delta_3 \alpha \\
0 & \rho a^2 \delta_1 \alpha & \rho a^2 \delta_2 \alpha & \rho a^2 \delta_3 \alpha & q_\alpha
\end{bmatrix}.
\]  

(16)

The five eigenvalues are

\[
c_1 = \langle n, q \rangle - a, \quad c_2 = c_3 = c_4 = \langle n, q \rangle, \quad c_5 = \langle n, q \rangle + a.
\]  

(17)

We can easily check that there are three linearly independent left (or right) eigenvectors corresponding to the triple eigenvalue \( \langle n, q \rangle \) so that the system (12 - 14) is hyperbolic.
Bicharacteristic formulation of the Euler equations

The left and right eigenvectors corresponding to the eigenvalue $c_5$ can be chosen to be

$$
\ell = (0, n_1, n_2, n_3, \frac{1}{\rho a}), \quad r = (\rho/a, n_1, n_2, n_3, \rho a).
$$

The characteristic partial differential equation corresponding to this eigenvalue is

$$
\tilde{Q} \equiv \phi_t + \langle q, \nabla \phi \rangle + a|\nabla \phi| = 0.
$$

The ray equations for $x$ and $n$ become

$$
\frac{dx}{dt} = q + na
$$

and

$$
\frac{dn}{dt} = -La - n_\beta Lq_\beta.
$$
Bicharacteristic formulation of the Euler equations

Multiplying the equations in (12)-(14) by components of \( \ell \) and adding the results, we derive the compatibility condition on the characteristic surface as

\[
\alpha \frac{d\rho}{dt} + \rho \langle n, \frac{dq}{dt} \rangle + \rho a \langle L, q \rangle = 0,
\]

where \( \frac{d}{dt} = \frac{\partial}{\partial t} + \langle q + an, \nabla \rangle \). This is the form of the compatibility condition for the Euler equations (12)-(14) for the characteristic velocity \( c_5 \).
Comment on the compatibility condition

- The derivatives in the compatibility condition (1.15) on a characteristics surface $\Omega$ are so grouped that each group represents a tangential derivative on $\Omega$.
- The derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_\alpha \frac{\partial}{\partial x_\alpha}$ along a ray is the time derivative along a bicharacteristics i.e., tangential to $\Omega$.
- Each of the derivatives $\tilde{\partial}_j = \ell_i \left( B_{ij}^{(\alpha)} - \chi_\alpha A_{ij} \right) \frac{\partial}{\partial x_\alpha}$, $(j = 1, 2, \cdots, n)$ operating on $u_j$ in the second term represents tangential derivatives not only on $\Omega$ but also on $\Omega_t$. 
Comment on the equations in bicharacteristics theorem

- We note that $|\mathbf{n}| = 1$ so that only $m - 1$ components of $\mathbf{n}$ are independent and we can show that only $m - 1$ equations in (7) are independent.
- For a linear hyperbolic system, the ray equations (6) and (7) decouple from (9) and hence can be solved to give rays.
- For a quasilinear system (1), the matrices $A$ and $B^{(\alpha)}$ depend on $\mathbf{u}$ and hence the terms on the right sides of (6) and (7), when evaluated, would contain $\mathbf{u}$ and $\frac{\partial \mathbf{u}}{\partial \eta^\alpha_\beta}$. 
Comment on the equations in bicharacteristics theorem

- In this case the system (6), (7) and (9) of $2m$ equations in $2m + n - 1$ quantities $x, n$ (only $n - 1$), $u$ is an under-determined unless $n = 1$

- However, this system is useful in high frequency approximation, leading to the weakly nonlinear ray theory (WNLRT) or the shock ray theory.

- For a hyperbolic system, we get a complete system when we consider $n$ compatibility conditions corresponding to $n$ families of characteristics fields.
Comment on the equations in bicharacteristics theorem

- One important use is in development of numerical methods, namely characteristic Galerkin method (started by Morton (2000) and collaborators following Prasad and collaborators (1982)), a topic of active research today.

- We present an example, equivalent to the wave equation with variable speed in a heterogeneous medium in two space dimensions and derive all equations in the bicharacteristic theorem explicitly.

- In example, we shall see compatibility conditions of all characteristic families take part.
Mathematical Model Linear Heterogeneous Medium

- Considers Euler equations and small perturbations of steady state $\rho_0, u_0 = 0, v_0 = 0, p_0$, where $\rho, u, v, p$ denote respectively the density, $x, y$-components of velocity and pressure.
- It turns out from continuity equation $\rho_0 = \rho_0(x, y)$ and from momentum equations that $p_0$ has to be a constant.
- The acoustic waves are then governed by a first order system (A., Kraft, Lukacova, Prasad, 2009).
Mathematical Model Linear Heterogeneous Medium — continued

\[ H_t + (f_1(u))_x + (f_2(u))_y = 0, \quad (23) \]

where

\[
H = \begin{bmatrix}
p \\
\rho_0 u \\
\rho_0 v
\end{bmatrix}, \quad f_1(u) = \begin{bmatrix}
a_0^2 \rho_0 u \\
p \\
0
\end{bmatrix}, \quad f_2(u) = \begin{bmatrix}
a_0^2 \rho_0 v \\
0 \\
p
\end{bmatrix}
\]

and \( a_0 = \sqrt{\gamma p_0 / \rho_0} \) denotes the wave speed and

\[
u = \begin{bmatrix}
p \\
u \\
v
\end{bmatrix}.
\]

Note that \( \rho_0 = \rho_0(x, y), \ p_0 \equiv \text{const} \) and \( a_0 = a_0(x, y) \).
Governing Equations and Eigenvalues

In differential form this reads

\[ v_t + A_1 v_x + A_2 v_y = 0, \]  

(24)

where

\[ u = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \gamma p_0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & \gamma p_0 \\ 0 & 0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 \end{bmatrix}. \]

Three eigenvalues of this system are

\[ c_1 = -a_0, \quad c_2 = 0, \quad c_3 = a_0, \]  

(25)

where \( a_0 > 0 \).
Eigenvectors

Take unit normal \((n_1, n_2) = (\cos \theta, \sin \theta)\), then left eigenvectors are

\[
\begin{align*}
\mathbf{l}_1 &= \frac{1}{2} \left( -\frac{1}{a_0 \rho_0}, \cos \theta, \sin \theta \right), \\
\mathbf{l}_2 &= (0, \sin \theta, -\cos \theta), \\
\mathbf{l}_3 &= \frac{1}{2} \left( \frac{1}{a_0 \rho_0}, \cos \theta, \sin \theta \right).
\end{align*}
\] (26)

Right eigenvectors are

\[
\begin{align*}
\mathbf{r}_1 &= \begin{bmatrix} -a_0 \rho_0 \\ \cos \theta \\ \sin \theta \end{bmatrix}, \\
\mathbf{r}_2 &= \begin{bmatrix} 0 \\ \sin \theta \\ \cos \theta \end{bmatrix}, \\
\mathbf{r}_3 &= \begin{bmatrix} a_0 \rho_0 \\ \cos \theta \\ \sin \theta \end{bmatrix}.
\end{align*}
\] (27)
Bicharacteristic Theorem for First Characteristic Field

Bicharacteristic equations are

\[
\frac{dx}{dt} = -a_0(x, y) \cos \theta, \quad \frac{dy}{dt} = -a_0(x, y) \sin \theta, \\
\frac{d\theta}{dt} = -a_{0x} \sin \theta + a_{0y} \cos \theta.
\]  
\tag{28}

The transport equation or compatibility condition is

\[
\frac{dp}{dt} - z_0 \cos \theta \frac{du}{dt} - z_0 \sin \theta \frac{dv}{dt} + z_0 S = 0,
\]  
\tag{29}

where \( z_0 = a_0 \rho_0 \) and

\[
S := a_0 \left\{ u_x \sin^2 \theta - (u_y + v_x) \sin \theta \cos \theta + v_y \cos^2 \theta \right\}
\equiv a_0 \left( -\sin \theta \frac{\partial u}{\partial \lambda} + \cos \theta \frac{\partial v}{\partial \lambda} \right), \quad \frac{\partial}{\partial \lambda} \text{ is defined below.}
\]  
\tag{30}
Geometrical and Physical Interpretation

- The operator
  \[
  \frac{d}{dt} = \frac{\partial}{\partial t} - a_0 \cos \theta \frac{\partial}{\partial x} - a_0 \sin \theta \frac{\partial}{\partial y}
  \]
  \[\text{(31)}\]
  represents time rate of change along the backward bicharacteristics.

- The operator
  \[
  \frac{\partial}{\partial \lambda} = - \sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}
  \]
  \[\text{(32)}\]
  represents a tangential derivative on wavefront $\Omega_t$. 
Geometrical and Physical Interpretation

- (16) shows that wavefront normal turns due to gradient of sound velocity $a_0$ along $\Omega_t$. Rays are not straight lines.

- We can write $S$ as

$$S = \tilde{\partial}_1 u + \tilde{\partial}_2 v$$

where $\tilde{\partial}_1 = -a_0 \sin \theta \frac{\partial}{\partial \lambda}$ and $\tilde{\partial}_2 = a_0 \cos \theta \frac{\partial}{\partial \lambda}$.

- (18) shows that evolution of a combination of $u$, $v$, and $p$ along the bicharacteristic depends not only the value of the combination at a base point $P_0$ but also on the tangential derivatives of $u$ and $v$ along the wavefront at the base point.
Backward Moving Wavefronts and Rays of a Wave Equation, $a(x, y) = a_0 + a_1(x - x_P) + a_2(y - y_P)$. Wavefront converges to a point.
Derivation of Integral form of Evolution Equation

Let $\omega = \theta(t_{n+1}) \in [0, 2\pi]$ is a parameter such that $\omega = \text{const.}$ represents a particular bicharacteristic starting from the vertex $P = (x, y, t_{n+1} = t_n + \Delta t)$.

A solution representing a backward bicharacteristic starting from $P = (x, y, t_n + \Delta t)$ is represented as $x = x(t, \omega), \ y = y(t, \omega), \ \theta = \theta(t, \omega)$.

Let $Q_1 = Q_1(x(t_n), y(t_n), t_n), \ \tilde{Q}_1 = \tilde{Q}_1(x(\tau), y(\tau), \tau)$ be respectively the footpoints of a bicharacteristics on the planes $t = t_n$ and $t = \tau \in (t_n, t_{n+1})$. 
We integrate the transport equation (17) along the respective bicharacteristic (17) and take an integral average over the wavefronts.

Integrating (17) in time from $t_n$ to $t_{n+1}$ and using the integration by parts for the second and third terms yield
Derivation of Integral form of Evolution Equation — continued

\[ p(P) = p(Q_1) + \cos \omega(z_0 u)(P) - \cos \theta(z_0 u)(Q_1) \]

\[ - \int_{t_n}^{t_{n+1}} (z_0 a_0 x u)(\tilde{Q}_1) d\tau \]

\[ + \sin \omega(z_0 v)(P) - \sin \theta(z_0 v)(Q_1) \]

\[ - \int_{t_n}^{t_{n+1}} (z_0 a_0 y v)(\tilde{Q}_1) d\tau \]

\[ - \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1) d\tau. \]  (34)
Integrate (22) over $\omega \in [0, 2\pi]$ and divide by $2\pi$ to obtain

$$p(P) = \frac{1}{2\pi} \int_{0}^{2\pi} (p - z_0 u \cos \theta - z_0 v \sin \theta)(Q_1)d\omega$$

$$- \frac{1}{2\pi} \int_{0}^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 (a_{0x} u + a_{0y} v))(\tilde{Q}_1)d\tau d\omega$$

$$- \frac{1}{2\pi} \int_{0}^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1)d\tau d\omega. \quad (35)$$

This is the exact integral representation for $p$. 
2nd Family of Bicharacteristics: $\lambda = 0$

Bicharacteristic equations are

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0, \quad \frac{d\theta}{dt} = 0. \quad (36)$$

The compatibility condition is

$$z_0 \sin \theta \frac{du}{dt} - z_0 \cos \theta \frac{dv}{dt} + a_0(p_x \sin \theta - p_y \cos \theta) = 0. \quad (37)$$

Note that characteristic conoid degenerates into a single bicharacteristic line with $\theta(\tau) = \omega$

$$x = x_{n+1}, \quad y = y_{n+1}, \quad t = \tau \in (t_n, t_{n+1}). \quad (38)$$
2nd Family of Bicharacteristics: $\lambda = 0$ — continued

The footpoints of the bicharacteristics on the planes $t = t_n$ and $t = \tau \in (t_n, t_{n+1})$ are $Q_2 = (x_{n+1}, y_{n+1}, t_n)$, $\tilde{Q}_2 = \tilde{Q}_1(x(\tau), y(\tau), \tau)$ be respectively.

Integrating now (25) in time from $t_n$ to $t_{n+1}$ gives

$$
\sin \omega (z_0 u)(P) - \sin \omega (z_0 u)(Q_2) - \int_{t_n}^{t_{n+1}} \left( \frac{d}{dt} (\sin \theta z_0) u \right) (\tilde{Q}_2) d\tau
$$

$$
- \cos \omega (z_0 v)(P) + \cos \omega (z_0 v)(Q_2) - \int_{t_n}^{t_{n+1}} \left( \frac{d}{dt} (\cos \theta z_0) v \right) (\tilde{Q}_2) d\tau
$$

$$
+ \int_{t_n}^{t_{n+1}} \left( a_0 \left( \sin \omega p_x - \cos \omega p_y \right) \right) (\tilde{Q}_2) d\tau = 0. \quad (39)
$$

Note that the first two integrals disappear because $\frac{dz_0(x,y)}{dt} = 0$ and due to the ray equations (24).
Multiplying (28) by \( \sin \omega \) and integrating over \( \omega \) gives

\[
\pi(z_0 u)(P) - \pi(z_0 u)(Q_2) + \pi a_0(Q_2) \int_{t_n}^{t_{n+1}} p_x(\tilde{Q}_2) d\tau = 0.
\]  \hspace{1cm} (40)

This gives transport equation along single bicharacteristic curve (25) corresponding to convective wave gives \( u(x_{n+1}, y_{n+1}, t_{n+1}) \) purely in terms of values along this bicharacteristics is very weak information.
1st Family of Bicharacteristics: $\lambda = -a_0$ — continued

- We combine (28) with stronger information on the wavefront at $t_n$ of first family of characteristic.

Multiply (22) by $\cos \omega$ and integrate over $\omega$ to get

$$\pi z_0(P)u(P) = \int_0^{2\pi} \left( -p + z_0 u \cos \theta + z_0 v \sin \theta \right) (Q_1) \cos \omega d\omega$$

$$+ \int_0^{2\pi} \int_{t_n}^{t_{n+1}} \left( z_0 \left( a_0x u + a_0y v \right) \right) (\tilde{Q}_1) \cos \omega d\tau d\omega$$

$$+ \int_0^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1) \cos \omega d\tau d\omega.$$  (41)
Final Exact Integral Equation for $u, v, p$ at $P$

- Adding (22) and (29) and rearranging yields the integral equation for $u(P)$, we shall write this exact equation later.
- Analogously the exact integral representation for $v(P)$ can be derived.
- We do not need to consider the third family of characteristic compatibility condition corresponding to $c_3 = a_0$. This takes information in future i.e., $t > t_{n+1}$.
- We shall write all three exact integral equations for $u, v, p$ at $P$ on the next slides:
Integral Equation for $p(P)$ is

$$p(P) = \frac{1}{2\pi} \int_{0}^{2\pi} (p - z_0 u \cos \theta - z_0 v \sin \theta) (Q_1) d\omega$$

$$- \frac{1}{2\pi} \int_{0}^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 (a_{0x} u + a_{0y} v)) (\tilde{Q}_1) d\tau d\omega$$

$$- \frac{1}{2\pi} \int_{0}^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1) d\tau d\omega. \quad (42)$$
Integral Equation for $u(P)$

Integral Equation for $u(P)$ is

\[
\begin{align*}
    u(P) &= \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \left( -p + z_0 u \cos \theta + z_0 v \sin \theta \right) (Q_1) \cos \omega d\omega \\
    + \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \int_{t_n}^{t_{n+1}} z_0 \left( a_{0x} u + a_{0y} v \right) (\tilde{Q}_1) \cos \omega d\tau d\omega \\
    + \frac{1}{2} u(Q_2) - \frac{1}{2\rho_0(P)} \int_{t_n}^{t_{n+1}} p_x(\tilde{Q}_2) d\tau \\
    + \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1) \cos \omega d\tau d\omega.
\end{align*}
\]  
(43)
Integral Equation for $\nu(P)$

Integral Equation for $\nu(P)$ is

$$
\nu(P) = \frac{1}{2\pi z_0(P)} \int_0^{2\pi} (-p + z_0 u \cos \theta + z_0 \nu \sin \theta) (Q_1) \sin \omega d\omega
$$

$$
+ \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 (a_{0x} u + a_{0y} \nu)) (\tilde{Q}_1) \sin \omega d\tau d\omega
$$

$$
+ \frac{1}{2} \nu(Q_2) - \frac{1}{2\rho_0(P)} \int_{t_n}^{t_{n+1}} p_y(\tilde{Q}_2) d\tau
$$

$$
+ \frac{1}{2\pi z_0(P)} \int_0^{2\pi} \int_{t_n}^{t_{n+1}} (z_0 S)(\tilde{Q}_1) \sin \omega d\tau d\omega.
$$

(44)

We may treat all these three equations together as an extension of d’Alembert solution to multi-dimension, there solution is given directly. Here it is an integral equation.
Finite Volume Scheme

Divide a computational domain $\Omega$ into a finite number of regular finite volumes $\Omega_{ij} := [i\Delta x, (i + 1)\Delta x] \times [j\Delta y, (j + 1)\Delta y]$ for $i = 0, \ldots, M$, $j = 0, \ldots, N$

$$U_{ij}^0 = \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} U(\cdot, 0)d\Omega. \tag{45}$$

The update formula for the finite volume evolution Galerkin scheme is

$$U_{ij}^{n+1} = U_{ij}^n - \Delta t \frac{\Delta x}{\Delta t} \delta_{ij} \bar{f}_1^{n+1/2} - \Delta t \frac{\Delta y}{\Delta t} \delta_{ij} \bar{f}_2^{n+1/2}. \tag{46}$$

We evolve the cell interface fluxes $\bar{f}_k^{n+1/2}$ to $t_n + 1/2$ using the approximate evolution operator denoted by $E_{\Delta t/2}$ and average them along the cell interface $\mathcal{E}$

$$\bar{f}_k^{n+1/2} := \sum_j \omega_j f_k(E_{\Delta t/2} U^n(x_j^i(\mathcal{E}))), \quad k = 1, 2. \tag{47}$$

Here $x_j^i(\mathcal{E})$ are the nodes and $\omega_j$ the weights of the quadrature for the flux integration along the edges.
The building blocks of the FVEG scheme are

- **Step 1**: Polynomial reconstruction of the piecewise constant data using standard recovery procedures.
- **Step 2**: Discretise the flux integrals in the FV update using either Trapezoidal or Simpson rule.
- **Step 3**: Construct the local Mach cone at the quadrature nodes.
- **Step 4**: Evolve the data using the approximate evolution operator and compute fluxes at half time step.
- **Step 5**: Update the solution using the standard FV scheme.

**Remark**

*The FVEG method is a genuine multi-dimensional generalisation of Godunov’s REA algorithm.*
Smoothly varying wave speed

The computational domain is \([0, 1] \times [0, 1]\) and the initial conditions are

\[
p(x, y) = \sin(2\pi x) + \cos(2\pi y),
\]

\[
u(x, y) = 0,
\]

\[
v(x, y) = 0.
\]

The initial wave speed is

\[
a_0(x, y) = 1 + \frac{1}{4} (\sin(4\pi x) + \cos(4\pi y)).
\]

Periodic boundary conditions and final time is \(T = 1.0\).
Smoothly varying wave speed

Figure: Results with a smoothly varying wave speed
Radially symmetric wave speed

We model the wave propagation in a radially symmetric medium. The wave speed is

\[ a_0(x, y) = \begin{cases} 
0.175 & \text{if } r \leq 0.15, \\
0.350 & \text{if } 0.41 < r \leq 0.59, \\
0.275 & \text{if } 0.85 < r.
\end{cases} \]

The initial pressure is given by

\[ p(x, y) = \begin{cases} 
\bar{p}((r - 0.5)/0.18) & \text{if } |r - 0.5| < 0.18, \\
0 & \text{otherwise}.
\end{cases} \]

\( \bar{p} \) is a suitable polynomial.
Radially symmetric

Figure: The solution corresponding to radially symmetric wave speed $a_0$
Heterogeneous medium with discontinuous wave speed

Propagation of acoustic waves through a layered medium with a single interface. The piecewise constant wave speed is given as

\[ a_0(x, y) = \begin{cases} 
1.0 & \text{if } x < 0.5, \\
0.5 & \text{otherwise.} 
\end{cases} \]

The initial data are

\[ p(x, y) = \begin{cases} 
1.0 + 0.5(\cos(\pi r/0.1) - 1.0) & \text{if } r < 0.1, \\
0.0 & \text{otherwise.} 
\end{cases} \]

\[ u(x, y) = v(x, y) = 0.0. \]
Heterogeneous medium

Figure: The pressure isolines for the reflection problem
Thank You for Your Kind Attention!

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